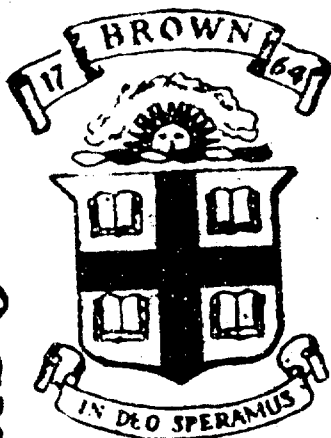


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MODE APPROXIMATIONS FOR IMPULSIVELY LOADED  
RIGID PLASTIC STRUCTURES

BY

J. B. MARTIN and P. S. SYMONDS

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IMPULSIVELY LOADED RIGID-PLASTIC STRUCTURES

by

J. B. Martin<sup>1</sup> and P. S. Symonds<sup>2</sup>

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Summary:

An attempt is made to provide a rational method of constructing one degree of freedom approximations for impulsively loaded metal structures which are analysed according to an elementary rigid-plastic theory. The approximation follows automatically from a chosen mode shape, and a criterion for determining good mode shapes is introduced.

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## 1. INTRODUCTION

The problem of the estimation of permanent deformations in ductile metal structures subjected to high intensity dynamic loading has received increasing attention over the past fifteen years. The problem is, in its entirety, one of great difficulty. Above the complexity of the problem of elastic structures subjected to transient loading, additional complicating factors which are of importance include the dissipation of energy in plastic work, elastic unloading from plastic states, work hardening, the dependence of yield stress upon rate of strain, geometry change and diverse other non-linear effects. Very few solutions to dynamic loading problems have recognized all these factors. More commonly, one or more of the factors are assumed to dominate the behavior of the structure, and all others are neglected or approximated.

Probably the most widely used approximation<sup>3</sup> is the replacement of the distributed mass of the structure by one or more point masses. This approximation has been used in conjunction with a variety of idealizations of the material behavior.

In fairly simple structures the actual distribution of mass, elastic stiffness and yield stress can be considered. Solutions for an elastic, perfectly plastic

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<sup>3</sup>See, for example:

N. M. Newmark, "A method of computation for structural dynamics," Proc. E. Mech. Div., A.S.C.E., 85, (EM3), p. 67, 1959.

C. H. Norris, et al. Structural Design for Dynamic Loads, McGraw-Hill Book Co., New York, N. Y., 1959.

material have been found<sup>4</sup>, but are difficult if the load magnitudes are much larger than those causing the yield stress to be reached. A variety of dynamic loading problems has been solved using what will be termed in this paper an elementary rigid-plastic theory. A representative but by no means complete list of papers is given.<sup>5</sup> The elementary rigid-plastic theory involves the following idealizations:

- (i) For the purpose of the dynamic loading problem under consideration a ductile material is represented by a rigid, perfectly plastic constitutive equation. All elastic effects are in consequence excluded.
- (ii) Geometry changes are assumed to be small, and the yield stress is assumed to be independent of the rate of strain.

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<sup>4</sup>For example:

H. H. Bleich and M. G. Salvadori, "Impulsive motion of elasto-plastic beams," Trans. A.S.C.E., 120, p. 499, 1955.

J. A. Seiler, B. A. Cotter, and P. S. Symonds, "Impulsive loading of elastic-plastic beams," J. App. Mech., 23, p. 515, 1956.

<sup>5</sup>E. H. Lee and P. S. Symonds, "Large plastic deformations of beams under transverse impact," J. App. Mech., 19, p. 308, 1952.

M. F. Conroy, "Plastic-rigid analysis of long beams under transverse impact," J. App. Mech., 19, p. 465, 1952.

P. S. Symonds, "Dynamic load characteristics in plastic bending of beams," J. App. Mech., 20, p. 475, 1953.

H. G. Hopkins and W. Prager, "On the dynamics of plastic circular plates," z. angew. Math. u Phys., 5, p. 317, 1954.

E. W. Parkes, "The permanent deformation of a cantilever struck transversely at its tip," Proc. Roy. Soc., A228, p. 462, 1955.

P. G. Hodge, Jr., "Impact pressure loading of rigid-plastic cylindrical shells," J. Mech. Phys. Sol., 3, p. 176, 1955.

The use of the elementary rigid-plastic theory is the subject of this paper. It is not within the scope of the report, however, to give a complete appraisal of the utility and applicability of this theory. It is entirely a valid method of approach, as are the other idealizations mentioned above, provided that care is taken to define the range where idealization and reality have something in common, and provided that experimental evidence confirms the theoretical predictions in this range. The elementary rigid-plastic theory, when applied to structures subjected to short duration, high intensity loading, has a range of validity bounded at one end by the requirement that the deformations should not be so large that geometry change effects are significant, and at the other end by the requirement that the energy of the disturbance should be large compared to the energy which could be stored elastically in the structure in order to justify the exclusion of elastic effects. These requirements are in some senses contradictory since the large disturbance needed to meet the energy requirement will tend to produce large deformations. Thus the extent of the possible range of validity of the elementary rigid-plastic theory must depend on the configuration and flexibility of the structure, and in some cases may not exist at all.

Experiments show that the rigid-plastic theory almost always requires corrections for the dependence of yield stress on strain rate, strain hardening and geometry changes. When these corrections are made, remarkably good agreement

has been achieved<sup>6</sup>, showing that the neglect of elastic effects is permissible when the total energy dissipated is much greater than maximum energy which could be stored elastically. In appropriate circumstances it is possible to make the corrections for strain rate sensitivity and for strain hardening by simply multiplying the static yield stress by a constant factor<sup>7</sup>; however, the "appropriate circumstance are not always obvious, and this method should be used with caution for a highly rate sensitive material such as mild steel.

In the authors' opinion the importance of the elementary rigid-plastic theory lies in its ability to provide, quickly and simply, an estimate of major deformations due to very large dynamic loads. Such an estimate provides a convenient basis for more refined analyses and calculations to include effects of strain rate sensitivity, finite deflections, and other effects when necessary.

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<sup>6</sup>S. R. Bodner and P. S. Symonds, "Experimental and theoretical investigation of the plastic deformation of cantilever beams subjected to impulsive loading," J. App. Mech., 29, p. 719, 1962.

E. H. Lee and S. J. Tupper, "Analysis of plastic deformation in a steel cylinder striking a rigid target," J. App. Mech., 21, p. 63, 1954.

T. J. Mentel, "The plastic deformation due to impact at a cantilever beam with an attached tip mass," J. A. M., 25, p. 515, 1958.

T. C. T. Ting and P. S. Symonds, "Impact of a cantilever beam with strain rate sensitivity," Proc. Fourth U. S. Nat. Congr. App. Mech., A. S. M. E., p. 1153, 1962.

A. J. Frick and J. B. Martin, "The plastic deformation of a bent cantilever," Tech. Rept. NSF GP1115/16 from Brown University to the National Science Foundation, April 1965.

<sup>7</sup>E. W. Parkes, "The permanent deformation of a cantilever struck transversely at its tip," Proc. Roy. Soc., A 228, p. 462, 1955.

This report will be limited to the discussion of one dimensional continuous structures made up of bars and rods, curved or straight, whose cross dimensions are small compared with their length. Generalized stresses (bending moments, axial force, etc.) and generalized strains (curvatures, axial strains) will be used in the analysis. The following description of the plastic flow rule for a rigid-perfectly plastic material follows Prager.<sup>8</sup>

Let the generalized stresses acting at a section be

$$Q_j \quad (j = 1, \dots, n)$$

and let the associated generalized strain rates be

$$\dot{q}_j \quad (j = 1, \dots, n).$$

The dimension of a particular component of generalized strain is such that the stress-strain product has the dimension of work per unit length. The stress-strain relation is written in terms of a yield function  $\Phi(Q_j)$  which must be convex and which must contain the origin. The generalized strain rates are given in terms of the generalized stresses as follows:

$$\dot{q}_j = \lambda < \Phi(Q_j) > \frac{\partial \Phi}{\partial Q_j} \quad (1)$$

where

$$< \Phi(Q_j) > = 1 \quad \text{when} \quad \Phi(Q_j) = 0$$

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<sup>8</sup>W. Prager, An Introduction to Plasticity. Addison Wesley Press, (Reading, Massachusetts), 1959.

$$\langle \Phi(Q_j) \rangle = 0 \quad \text{when} \quad \Phi(Q_j) < 0$$

$$\lambda \geq 0 \quad \text{but otherwise unspecified.}$$

Stresses such that

$$\Phi(Q_j) > 0$$

are not admitted.

The geometric interpretation of equation (1) is now well known. If the components of the stress  $Q_j$  are plotted as coordinates in an  $n$  dimensional space a stress point is defined. Similarly the convex function  $\Phi = 0$  may also be plotted as a closed surface in the stress space. If the stress point lies inside the surface  $\Phi = 0$ , the strain rate is zero. If the stress point lies on the surface, the magnitude of the strain rate vector is not specified. However, if  $\dot{q}_j$  is plotted in the stress space with the stress point as origin, the strain rate vector is required to have the direction of the exterior normal to the surface  $\Phi = 0$  at that point. A two dimensional representation of this interpretation is given in Fig. 1.

This geometrical approach demonstrates that if  $Q_j^*$  is any other admissible state of stress (Fig. 1), then

$$(Q_j - Q_j^*) \dot{q}_j \geq 0 \quad (2)$$

Further, if  $Q_j^*$  itself lies on the yield surface and is associated with strain rate  $\dot{q}_j^*$  (shown in Fig. 2), it is clear that by a second application of the concept demonstrated in equation (2),

$$(Q_j - Q_j^*) (\dot{q}_j - \dot{q}_j^*) \geq 0 \quad (3)$$



These inequalities have been discussed in a more general context by Hill<sup>9</sup> and Drucker.<sup>10</sup>

Inequality (3) may be used to show that the velocity history at any point in a structure following the application of high intensity time dependent loading is unique, provided that changes in geometry may be ignored and that certain continuity requirements are satisfied. This result has been discussed by Martin<sup>11</sup>, and is based upon earlier work by Drucker<sup>12</sup> which appears to have passed unnoticed by most workers in this field.

Suppose that a given structure is subjected to the following loading and boundary conditions: On length  $S_T$  of the body time dependent tractions  $T_i'$  ( $i = 1, 2, 3$ ) are specified, and on the remainder of the structure  $S_u$  time dependent velocities  $\dot{u}_i'$  are prescribed.  $S_T$  and  $S_u$ , which together comprise the whole structure, may themselves be time dependent. Further, let the velocities at time  $t = 0$  be given by  $v_i$ . These quantities define the problem. Assume now that two solutions can be found. First, velocities  $\dot{u}_i$  associated with accelerations  $\ddot{u}_i$ ,

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<sup>9</sup>R. Hill, "A variational principle of maximum plastic work in classical plasticity," Quart. J. Mech. App. Math., 1, p. 18, 1948.

<sup>10</sup>D. C. Drucker, "A more fundamental approach to stress-strain relations," Proc. 1st U. S. Congr. App. Mech., A.S.M.E., p. 487, 1951.

<sup>11</sup>J. B. Martin, "A note on the uniqueness of solutions for dynamically loaded rigid-plastic and rigid-visco-plastic continua," Tech. Rept. NSF GP1115/12 from Brown University to the National Science Foundation, September 1964.

<sup>12</sup>D. C. Drucker, "A definition of a stable inelastic material," J. App. Mech., 26, p. 101, 1959.

resses  $Q_j$ , strain rates  $\dot{q}_j$ . Secondly, velocities  $\dot{u}_i^*$ , accelerations  $\ddot{u}_i^*$ , stresses  $Q_j^*$ , strain rates  $\dot{q}_j^*$ . Both solutions satisfy all the field equations.

Consider the two solutions at some time  $t > t_0$ . Since the tractions, inertias and stresses of each solution are in internal and external equilibrium, the differences in these quantities are in equilibrium. Similarly, the differences in the strain rates and velocities are compatible. Further, the difference in the tractions vanishes on  $S_T$ , and the difference in displacements vanishes on  $S_u$ . In consequence, from the principle of virtual velocities,

$$- \int_{S_T} m (\ddot{u}_i - \ddot{u}_i^*) (\dot{u}_i - \dot{u}_i^*) dS = \int_S (Q_j - Q_j^*) (\dot{q}_j - \dot{q}_j^*) dS \quad (4)$$

where  $m$  is the mass per unit length of the structure. From (3), the right-hand side is non-negative. Hence, provided that the velocity at each point is a continuous function of time, we may write

$$\frac{d}{dt} \int_{S_T} m (\dot{u}_i - \dot{u}_i^*) (\dot{u}_i - \dot{u}_i^*) dS \leq 0 \quad (5)$$

It is seen that a non-negative quantity involving the velocities  $\dot{u}_i$ ,  $\dot{u}_i^*$  must decrease with time. At time  $t = 0$ , however,  $\dot{u}_i = \dot{u}_i^* = v_i$ , and the non-negative quantity is initially zero. It follows that  $\dot{u}_i = \dot{u}_i^*$  for all  $t > 0$ .

The restriction imposed by the requirement that the velocity at each point should be a continuous function of time is a serious one in general. It may readily be shown that when discontinuities are permitted the solutions to certain problems are

not unique (Ting<sup>13</sup>). In practice, however, a large class of problems where discontinuities do not occur is of interest. This is the class of structural problems where shear and axial strains are stipulated zero (bending and torsion strains are permitted). The velocity at each point will be a continuous function of time in these cases, and hence uniqueness follows. This report will be restricted to problems falling into this class.

Further, the report will be concerned only with impulsive loading problems. These problems may be characterized in exactly the same way as the general problem given above with the following restrictions:

- (i) The tractions applied to  $S_T$  will be taken to be zero.
- (ii) The velocities prescribed on  $S_u$  will be taken to be zero.
- (iii)  $S_u$  and  $S_T$  will be assumed to be time independent.

The impulsive loading problem is thus essentially one in which initial velocities are prescribed over the whole structure at time  $t = 0$ ; thereafter, no external forces do work on the structure.

No direct technique exists for determining the response of the structure to this (or any other more complex) form of loading. A solution must be sought on a trial and error basis. If a solution can be found which satisfies all the conditions (equilibrium, compatibility, yield and the stress-strain relation), the uniqueness result assures that this is the only solution for the velocity as a function of space and time.

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<sup>13</sup> T. C. T. Ting. Private communication, March 1965.

The response of a one-dimensional rigid-plastic structure to impulsive loading is characterized by two distinct phases of behavior. In the first phase, travelling "hinges" (discrete points at which plastic deformation occurs) associated with discontinuities in the acceleration field are found. In the second phase, deformation occurs without change of shape of the velocity field. The equations of the second phase may be written in terms of separate functions of time and space parameters, and hence are easily formulated.

In considering the elementary rigid-plastic solutions, Symonds<sup>14</sup> emphasized that in certain cases very simple closed form solutions of impulsive loading problems can be obtained, for example by momentum conservation equations in finite form. Examples were given<sup>15</sup> where slightly modified problems required numerical integration of the equations. Further examples have been given in the literature. Thus the problem of a mass striking a fixed end beam was solved in closed form by Parkes<sup>16</sup> but for the pin ended case the equations for the first phase have to be solved numerically (Ezra<sup>17</sup>).

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<sup>4</sup>P. S. Symonds, "Large plastic deformations of beams under blast loading," Proc. Second U. S. Nat. Congr. App. Mech., A.S.M.E., p. 505, 1954.

<sup>5</sup>P. S. Symonds, "Simple solutions of impulsive loading and impact problems of plastic beams and plates," Tech. Rept. UERD-3 from Brown University to the Norfolk Naval Shipyard under Contract N1895-1756 A, April 1955.

<sup>6</sup>E. W. Parkes, "The permanent deformation of an encasté beam struck transversely at any point in its span," Proc. Inst. Civ. Engrs. (London), 10, p. 266, 1958.

<sup>7</sup>A. A. Ezra, "The plastic response of a simply supported beam to an impact load at the center," Proc. 3rd U. S. Nat. Congr. App. Mech., A.S.M.E., p. 513, 1958.

A further example is given in Fig. 3. This is a fixed end beam subjected to impulsive loading so that the velocity is  $v_0$  in the region of length  $b$ . The complete solution can be easily obtained in closed form for any ratio  $b/l$ . However, if the ends are pinned rather than clamped a quite unpleasant numerical solution is required in order to obtain the three quantities required to define the motion in the first phase. This example will be discussed later in the paper.

Although the solution by numerical integration of a problem of elementary rigid-plastic theory is almost certain to be much less difficult than the solution of the same problem according to elastic-plastic theory, it may be tedious enough so that, for purposes of first estimates at least, one would welcome a simple approximate method. Such methods have frequently been used. The most common method is equivalent to the replacement of the structure by a mass-spring system of one degree of freedom. Matching the model to the actual problem has been done in various ad hoc ways. No attempt seems to have been made to generalize or systematize the various approximate methods.

This report will discuss a technique of approximation which is closely related to the matching of one degree of freedom models. Using an argument akin to that used to establish uniqueness, it will be shown that the solution to one impulsive loading problem may be approximated by the solution of a problem involving the same structure but different initial velocities.

To make use of this property we shall first give, for illustrative purposes, the solutions to two simple problems, and then show how approximations may be constructed for these and other problems. Finally, we shall discuss the relation between the two approaches to approximation, showing that the concepts of the previous

paragraph may be used to provide a rational method of matching one degree of freedom models to the structure.

## 2. SOME EXAMPLES OF SIMPLE SOLUTIONS

In this section we shall present two problems which can be very simply solved with a rigid-plastic theory. These solutions are given primarily to illustrate problems where the solution for the first phase, when travelling hinges occur, can be written in closed form

(i) The first problem considered will be the uniform cantilever subjected to transverse impulse on a point mass attached at the tip. The solution to this problem was obtained by Parkes<sup>7</sup>. The beam, with the relevant physical parameters, is shown in Fig. 4. At time  $t = 0$ , the tip mass acquires a velocity  $v$ . Shear strains are neglected in the analysis, and hence the bending moment is constrained to be between  $+M_0$  and  $-M_0$  for all shear force values.

In accordance with the trial technique for solution, a response is postulated and then checked. An assumed velocity field at time  $t > 0$  is shown in Fig. 5(a). The section AB of the beam rotates about B, and thus the bending moment at B must be zero. BC remains stationary, and thus for equilibrium of BC [Fig. 5(b)] the shear force in BC must be zero and the bending moment must be  $M_0$ . Certainly this distribution of velocity and strain rate is kinematically admissible, and the stress-strain relation has been satisfied.

We give the solution obtained by Parkes<sup>7</sup> by writing conservation of momentum equations for the beam. Since no external forces act on the beam the linear momentum of the beam cannot be changed: hence at time  $t$

$$G\dot{z} + \frac{Mx\dot{z}}{2} = Gv \quad (6)$$

Angular momentum about a line through A perpendicular to both the beam center line and the direction of motion will be considered. The initial angular momentum about this line is zero; hence at time  $t$

$$\left(\frac{mx}{2}\dot{z}\right)\frac{x}{3} = M_o t \quad (7)$$

These equations give  $\dot{z}$  and  $t$  in terms of the parameter  $x$ . Introducing the dimensionless parameters

$$\gamma = ml/2G, \quad \xi = x/l,$$

(6) and (7) become

$$\left(\frac{\dot{z}}{v}\right) = \frac{1}{1 + \gamma\xi} \quad (8a)$$

$$\frac{M_o t}{Glv} = \frac{1}{3} - \frac{\gamma\xi^2}{(1 + \gamma\xi)} \quad (8b)$$

$\xi$  increases monotonically to give increasing  $t$ , and the solution remains valid until  $\xi = 1$  or the hinge reaches C. Thereafter the beam rotates as a rigid body about C, and the behavior enters the second phase. The shear force at C is no longer zero; however, angular momentum about C gives a relation between  $\dot{z}$  and  $t$  directly:

$$M_o t = \frac{\gamma}{3} - \frac{Glv}{(1 + \gamma)} + Glv \frac{(1 + \frac{2}{3}\gamma)}{(1 + \gamma)} - Gl\dot{z} (1 + \frac{2}{3}\gamma) \quad (9)$$

Rearranging

$$\frac{M_o t}{Glv} = 1 - \left(\frac{\dot{z}}{v}\right) \left(1 + \frac{2}{3}\gamma\right) \quad (10)$$



This relation applies until  $\dot{z} = 0$ .

The equations derived here can be shown to lead to a distribution of bending moments which at all times satisfies the yield condition, and hence this is the solution to the problem. A closed form solution for the final displacement  $\delta$  of the tip mass may be obtained by integrating the tip velocity equations:

$$\frac{M_o \delta}{\frac{1}{2} G l v^2} = \frac{1}{3(1 + \gamma)} + \frac{2}{3\gamma} \log(1 + \beta) \quad (11)$$

(ii) As a second example, consider a uniform beam subjected to impulsive loading over a length  $b$  symmetric about the midpoint, so that the initial velocity is  $v_o$  on this section and zero in the remaining length  $(l - b)$  [Fig. 6(a)]. The yield condition is taken to be the same as that in the previous example. For  $b = l$  the solution was given by Symonds for clamped end and pin end conditions<sup>14, 15</sup>. In that case it is not difficult to obtain solutions in closed form. In practice, if the ends are constrained against axial motion, the effects of the constraints tend to dominate when the deflection exceeds magnitudes of the order of the beam depth. These effects have been considered<sup>18</sup>, but they lie beyond the scope of the present discussion.

For  $b/l \leq 1$ , if the ends are built in a closed form solution can be simply derived. Fig. 6(c) indicates the trial velocity configuration for an instant soon after motion begins. The sections AP and QC have zero shear force and bending moment  $-M_o$  and  $M_o$ , respectively. The plastic boundary points are at

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<sup>18</sup>P. S. Symonds and T. J. Mentel, "Impulsive loadings of plastic beams with axial constraints," J. Mech. Phys. Sol., 6, p. 186, 1958.

$$x = \xi_1(t)l$$

and

$$x = \xi_2(t)l,$$

and these define the motion in this stage. They can be found by writing the two equations of momentum conservation, namely of linear momentum and angular momentum about any point in AP or QC. Omitting details, the following results are found:

$$\xi_1^2 = \xi_2^2 = \frac{12 M_o t}{m l^2 v_o} \quad (12)$$

This stage ends when the hinge at P reaches the support point or that at Q reaches the midpoint. If  $b < l/2$ , the subsequent stage involves the pattern shown in Fig. 6(d), the unknown quantities defining the motion now being the midpoint velocity  $v(t)$  and  $\xi_1(t)$ . Momentum equations enable us to find these as follows:

$$\xi_1 = \frac{b}{4l} + \frac{12 M_o t}{m b l v_o} \quad (13a)$$

$$\frac{4 v_o}{v} = 3 + \frac{48 M_o t}{m b^2 v_o} \quad (13b)$$

These equations hold for times  $t$  such that

$$1 \leq \frac{48 M_o t}{m b^2 v_o} \leq \frac{2l}{b} - 3 \quad (14)$$

The latter time corresponds to the instant when the hinge at P reaches end A of the beam. The two stages treated above comprise the first phase of the motion.

In the second phase there are stationary plastic hinges at A and C. Each half beam rotates as a rigid bar. The midpoint velocity  $v$  is found, by writing an angular momentum equation, to be given by

$$\frac{2l^2}{b^2} \frac{v}{v_o} = \frac{6l}{b} - 3 - \frac{48 M_o t}{mb^2 v_o} \quad (15)$$

The beam comes to rest when  $t = t_f$ , where

$$\frac{48 M_o t_f}{mb^2 v_o} = \frac{6l}{b} - 3 \quad (16)$$

The deflections in the various stages are easily found by integrating the velocities written above. The result for the final midpoint displacement is

$$u_f = \frac{mb^2 v_o}{12 M_o} \left[ \frac{1}{4} + \ln \frac{l}{2b} + 1 \right], \quad b < \frac{l}{2} \quad (17)$$

The three terms show respectively the contributions from the three stages of motion.

Throughout the above analysis the yield condition is everywhere satisfied. Since the equations of dynamics, the boundary conditions and the initial velocity distribution are all satisfied, the result is the exact solution (according to the elementary rigid-plastic theory). The above results apply for

$$2b/l < 1.$$

A similar analysis for

$$2b/l \geq 1$$

gives the result

$$u_f = \frac{mb^2 v_o}{48 M_o} \left[ \frac{6l}{b} - \frac{l^2}{b^2} - 3 \right] \quad (18)$$

When the ends of the beam are pinned rather than built in, the solution for general  $b/l < 1$  cannot be obtained in the simple manner used above. The velocity pattern and free body diagram for the first stage are shown in Figs. 7(b) and 7(d). These enable the initial velocity conditions and all other requirements to be satisfied. There are now four unknowns; in addition to  $\xi_1$  and  $\xi_2$  the velocity  $v_1$  at the hinge point P and the reaction force  $R_A$  must be found. The four equations in these unknowns can be written in various ways and will be omitted here. Velocities are continuous at the hinge points P and Q, but the accelerations are not. The numerical integrations required to obtain the complete solution are tedious. They have been carried through for  $b/l = 0.2$  and  $b/l = 0.7$ .<sup>19</sup>

The two examples given show just how easily some solutions in the elementary rigid-plastic theory may be obtained. The case discussed in the previous paragraph shows, on the other hand, how a small change in initial or boundary conditions may enormously complicate the analysis. The need for a simpler way of obtaining an approximation to the complete solution in such cases is apparent.

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<sup>19</sup>G. Gangopadhyay. M. Sc. Thesis, Brown University, 1964. Further calculations on this problem have been carried out by K. Vashi.

### 3. ONE DEGREE OF FREEDOM APPROXIMATIONS

A one degree of freedom model appropriate to a rigid-plastic material is shown in Fig. 8(a). The spring force-spring displacement characteristic is as shown in Fig. 8(b); essentially motion of the mass is resisted by a constant force  $S$ . If the spring is initially undeformed and has initial velocity  $\dot{u}_0$ , it is clear that the velocity  $\dot{u}$  at time  $t$  is given by

$$G\dot{u} = G\dot{u}_0 - St \quad \text{for } \dot{u} \geq 0$$

or

$$\frac{\dot{u}}{\dot{u}_0} = 1 - \frac{St}{G\dot{u}_0} \quad \text{for } \frac{\dot{u}}{\dot{u}_0} \geq 0 \quad (19)$$

Thus the velocity of the mass decreases with time, coming to rest in time  $t_f = G\dot{u}_0/S$ .

The final displacement is

$$u_f = \frac{1}{2} G\dot{u}_0^2/S.$$

In order to apply these results to, for example, a beam problem, suitable values must be found for the quantities  $\dot{u}$ ,  $S$  and  $G$ . These quantities may be referred to as the equivalent velocity, equivalent resisting force and equivalent mass respectively.

Consider, for example, the beam shown in Fig. 9(a). This case was discussed in detail in the previous section. A one degree of freedom approximation will be given for this example in order to demonstrate a method of finding the equivalent quantities.

In the case of the spring (Fig. 8) the limiting value of the resisting force may be found by statically applying an external force to the spring and finding the value of the external force required to initiate flow. This external force and the resisting force will have equal magnitudes. This simple idea provides one means of finding an equivalent force: Apply static loading to the beam and take the limiting value of the load parameter as the equivalent resisting force.

In this case choose the loading pattern shown in Fig. 9(b). Only half the beam is shown because of the symmetry of the system. Statics shows that the limiting value of  $P$  is given by

$$P_c = \frac{8 M_o}{2l - b} \quad (20)$$

Steady flow will occur in the beam when  $P$  reaches  $P_c$ . The flow field associated with  $P_c$  suggests itself as a reasonable means of choosing the equivalent velocity; when impulsive loading is applied, it will be assumed that the velocity field has the same shape as the velocity field shown in Fig. 9(c). (Notice that the velocity field for steady flow under loads  $P_c$  is not unique. The field given is a possible configuration). It remains, therefore, to determine the amplitude of the velocity field as a function of time.

One method of determining the acceleration is to write the angular acceleration equation for half the beam about the hinge support. If  $\dot{u}^*$  and  $\ddot{u}^*$  are the velocity and acceleration of the center of the beam,

$$\left(\frac{ml^3}{24}\right) \left(\frac{\ddot{u}^*}{l}\right) = -2M_o \quad (21)$$

Integrating and substituting from equation (20)

$$\dot{u}^* = v^* - \frac{6(2l-b)}{ml^2} P_c t \quad (22)$$

where  $v^*$  is the initial value of  $\dot{u}^*$ . The equivalent mass, in analogy with equation (19), is given by

$$G = \frac{ml^2}{6(2l-b)} \quad (23)$$

The maximum central displacement is given by

$$u_f = \frac{ml^2 \dot{u}_o^2}{12 P_c (2l-b)} \quad (24)$$

It is now necessary to ascribe a value to  $v^*$ . This may be done in a variety of ways; it is possible to match the initial energy of the actual problem and the one degree of freedom system, or to match either the linear or angular momentum of two systems. Computations based on these three criteria show that the best approximation is obtained when angular momentum of half the beam about the supporting hinge is balanced. Thus we put

$$\left(\frac{ml^3}{24}\right)\left(\frac{2v^*}{l}\right) = \left(\frac{mbv_o}{2}\right)\left(\frac{l}{2} - \frac{b}{4}\right)$$

i.e.

$$v^* = \frac{3b}{2} \frac{(2l-b)}{l^2} v_o \quad (25)$$

Substituting into (24) and rearranging in the form given in equation (18), the approximate central displacement is given by

$$u_f^* = \frac{mb^2 v_o^2}{12 M_o} \left[ \frac{9}{32} - \frac{(2l - b)^2}{l^2} \right] \quad (26)$$

The derivation of this expression involves many separate decisions which appear natural but are also arbitrary. If the method is to be applied to find approximate solutions of problems which have not been fully solved it is desirable that the application of the method should require fewer arbitrary decisions. A more sophisticated approach using the principle of virtual velocities will eliminate most of these decisions.

This method is based on the premise that the choice of a mode shape is the most important part of the approximation process. Suppose that, for a given structure, the solution is assumed to be of the form

$$\dot{u}_i^* (S, t) = S_i (S) T (t) \quad (27)$$

where  $S_i$  is a vector valued function of the space variable  $S$  alone, and  $T$  is a scalar function of time alone. Solutions of this form will be referred to as mode solutions. Some freedom is permitted in arranging the magnitude and dimensions of the functions  $S_i$  and  $T$ . For convenience we shall assume that  $T$  is dimensionless and that it has the value unity when  $t = 0$ .  $S_i$  will thus have the dimensions of velocity; it will in fact be the initial velocity distribution. For simplicity in solution we shall write (27) as

$$\dot{u}_i^* (S, t) = v_i^* (S) T (t) \quad (28)$$



If the solution to an impulsive loading problem is to be assumed to be of the form of (28) it is clearly desirable that  $v_i^*(S)$  should satisfy the kinematic boundary conditions.

We shall show first that  $T(t)$  must be a linear function of time if the intern energy dissipation rate associated with the assumed velocity field is to be equal to the rate of change of kinetic energy. The rate of dissipation of the kinetic energy  $K$  is given by

$$\begin{aligned} \frac{dK}{dt} &= \frac{d}{dt} \int_S \frac{m}{2} \dot{u}_i^* \dot{u}_i^* dS \\ &= \frac{d}{dt} (T^2) \int_S \frac{m}{2} v_i^* v_i^* dS \\ &= T \frac{dT}{dt} \int_S m v_i^* v_i^* dS \end{aligned} \quad (29)$$

The strains  $\dot{q}_j$  may be derived from the velocities  $\dot{u}_i^*$ . Let the initial values of these strains, derived from  $v_i^*$ , be  $\bar{q}_j$ . These strain quantities will be related in the same way as the velocities

$$\dot{q}_j^* = \bar{q}_j T \quad (30)$$

When the strain component  $\dot{q}_j^*$  is plotted as a vector in the stress space (as in Fig. it is clear that  $\dot{q}_j^*$  changes only in magnitude as time progresses. It is, however, the direction of  $\dot{q}_j^*$  which determines the associated stress vector  $Q_j^*$ . (Although  $Q_j$  may not always be uniquely determined, the dissipation rate is unique for a

given  $\dot{q}_j$ .) Thus  $Q_j^*$  does not change with time, and the dissipation rate  $D^*$  is given by

$$D^* = Q_j^* \dot{q}_j^* = Q_j^* \bar{\dot{q}}_j T = \bar{D} T \quad (31)$$

where  $\bar{D} = Q_j^* \bar{\dot{q}}_j$  is the initial dissipation rate.

We now equate the rate of change of kinetic energy and the rate of change of internal energy, given by the total energy dissipation rate with a negative sign.

$$T \frac{dT}{dt} \int_S m v_i^* v_i^* dS = -T \int_S \bar{D} dS$$

$$\frac{dT}{dt} = - \frac{\int_S \bar{D} dS}{\int_S m v_i^* v_i^* dS} = -\lambda \quad (32)$$

$\lambda$  is a constant, defined by (32), which can be readily determined from the initial velocity distribution. Solving equation (32) for  $T$ , with the requirement that  $T = 1$  when  $t = 0$ , we obtain

$$T = (1 - \lambda t) \quad (33)$$

The time  $t_f$  which elapses before motion in the assumed mode ceases is given simply by

$$t_f^* = \frac{1}{\lambda} \quad (34)$$

It is still necessary to determine the initial velocity field  $v_i^*$ , since only the mode shape and not the initial amplitude has been assumed. Suppose that a mode

shape  $\phi_i$  is assumed. Then

$$v_i^* = \alpha \phi_i \quad (35)$$

and it is necessary to choose the amplitude  $\alpha$ . The actual initial velocity distribution will be taken to be  $v_i$ .

A rational method of determining  $\alpha$  may be developed by assuming that one degree of freedom model is subjected, at time  $t = 0$ , to an impulse distribution  $mv_i$ , where  $m$  is the mass per unit length of the structure. The velocity with which the model begins to move may be then taken to be  $v_i^*$ .

This problem may be handled by means of the principle of virtual velocities.  $\phi_i$  is a kinematically admissible time independent velocity field. Let the associated strain rates be  $\dot{q}_j \phi_j$ . Suppose now that the model is subjected to forces  $P_i$  which over the time interval  $0 \leq t \leq \tau$ , causing accelerations  $\ddot{u}_i$  and stresses  $Q_j$ . For in the interval

$$\int_S P_i \phi_i dS - \int_S m \ddot{u}_i \phi_i dS = \int_S Q_j \dot{q}_j \phi_j dS \quad (36)$$

Integrate from time  $t = 0$  to  $t = \tau$ . Let the initial velocities of the model be zero and the final velocities  $v_i^*$ .

$$\int_S \left( \int_0^\tau P_i dt \right) \phi_i dS = \int_S m v_i^* \phi_i dS + \int_0^\tau dt \int_S Q_j \dot{q}_j \phi_j dS \quad (37)$$

Advantage has been taken of the fact that  $\phi_i$  is time independent. We now suppose

that  $\tau \rightarrow 0$  and that

$$\int_0^\tau P_i dt \rightarrow mv_i.$$

$v_i^*$  becomes the initial mode velocity and the term of the right of (37), representing a measure of work done in the time interval, will vanish. Finally

$$\int_S mv_i \phi_i dS = \int_S mv_i^* \phi_i dS \quad (38)$$

Substituting from (35) this expression may be written either as

$$\int_S mv_i v_i^* dS = \int_S mv_i^* v_i^* dS \quad (39a)$$

or

$$a = \frac{\int_S mv_i \phi_i dS}{\int_S m \phi_i \phi_i dS} \quad (39b)$$

Thus, referring back to equation (28), the velocities for a mode solution can be written in terms of an initial velocity field and a time function; the time function [equations (33) and (34)] and the initial velocity field can be determined completely in terms of an assumed mode shape  $\phi_i$  and the physical properties of the structure.

The application of this method will be illustrated by the case of the beam shown in Fig. 9. Assume a mode shape of the same form as shown in Fig. 9(c) i.e.

of the same form as was used in the previous approximation. Thus  $\phi$  is given by

$$\phi = \frac{2x}{l} \quad 0 \leq x \leq l/2 \quad (40a)$$

and

$$v^* = \frac{2x}{l} \dot{u}_0^* \quad 0 \leq x \leq l/2 \quad (40b)$$

The initial conditions for the actual problem (Fig. 9a) are given by

$$v_i \rightarrow 0 \quad 0 \leq x \leq (l-b)/2 \quad (41)$$

$$v_i \rightarrow v_0 \quad (l-b)/2 \leq x \leq l/2$$

To determine  $\dot{u}_0$  we use equation (39b). This gives

$$\begin{aligned} \dot{u}_0^* &= \frac{2 \int_{\frac{l-b}{2}}^{l/2} m v_0 \left(\frac{2x}{l}\right) dx}{2 \int_0^{l/2} m \left(\frac{2x}{l}\right)^2 dx} \\ &= \frac{3}{2} \frac{v b}{l^2} (2l - b) \end{aligned} \quad (42)$$

Comparing this result with equation (25), it is seen that the process carried out is equivalent to matching the initial angular momentum of half the beam. The method has an advantage, however, in that it may be easily applied in situation where angular momentum cannot be clearly understood. The time  $t_f^*$  which elapses

the model comes to rest is given by substitution into equations (32) and (34)

$$t_f^* = \frac{2 \int_0^{l/2} m \left(\frac{2x}{l}\right)^2 (\dot{u}_0^*)^2 dx}{4M_0 \left(\frac{2\dot{u}_0^*}{l}\right)}$$

$$= \frac{1}{8} \frac{m l^2 v_o^2}{M_0} \left[ 1 - \left(1 - \frac{b}{l}\right)^2 \right] \quad (43)$$

The initial velocity at the center of the beam is  $\dot{u}_0^*$ , and this velocity changes linearly with time. Hence the central displacement of the model is given by

$$u_f^* = \frac{\dot{u}_0^* t_f^*}{2}$$

$$= \frac{1}{12} \frac{m b^2 v_o^2}{M_0} \left[ \frac{9}{16} \left(2 - \frac{b}{l}\right)^2 \right] \quad (44)$$

Comparing this result with equation (26) which gives the displacement obtained from the earlier model, we find that the second approximation is twice as large as the first. This is a substantial difference. Without consulting the correct answer given in §2 of the paper, we would expect the second result to be more reliable than the first, since it involves fewer decisions on the behavior of the model. However, this is not necessarily true. Further, the choice of a mode shape itself is an arbitrary decision. Alternative mode shapes for the problem under discussion

are shown in Fig. 10. It may be possible that these shapes would give better approximations than Fig. 9(c), at least for certain values of  $b/l$ .

If the technique used to obtain the second approximation is to be useful in problems where the exact answer is not known, some method must be found which can differentiate between a good choice of mode shape and a poor choice. In the following sections we shall attempt to show how the ideas used in establishing uniqueness for the problems under discussion can be used to make such a differenti

#### 4. CONVERGING SOLUTIONS

Using an argument closely related to that used to prove uniqueness, a relation between the solutions to two independent impulsive loadings (or initial velocity distributions) on the same structure can be written, provided that the boundary conditions are the same.

We consider two identical structures with identical boundary conditions (i. e. velocities prescribed zero on  $S_u$ , tractions prescribed zero on  $S_T$ ). Let the initial velocities be respectively  $v_i(S)$  and  $v_i^*(S)$ , and let the solutions be respectively  $\dot{u}_i(S, t)$ ,  $\ddot{u}_i$ ,  $\dot{q}_j$ ,  $Q_j$  and  $\dot{u}_i^*(S, t)$ ,  $\dot{u}_i^*$ ,  $\dot{q}_j^*$ ,  $Q_j^*$ . As in equation (4), we may write,

$$-\int_S m (\ddot{u}_i - \ddot{u}_i^*) (\dot{u}_i - \dot{u}_i^*) dS = \int_S (Q_j - Q_j^*) (\dot{q}_j - \dot{q}_j^*) dS \geq 0 \quad (45)$$

or

$$\frac{d\Delta}{dt} \leq 0 \quad (46a)$$

where

$$\Delta = \int_S \frac{m}{2} (\dot{u}_i - \dot{u}_i^*) (\dot{u}_i - \dot{u}_i^*) dS \quad (46b)$$

The initial value of  $\Delta$ , at time  $t = 0$ , is given by

$$\Delta^0 = \int_S \frac{m}{2} (v_i - v_i^*) (v_i - v_i^*) dS \quad (47)$$

It is clear from (46b) that  $\Delta$  is a non-negative scalar function of time, and, from



(46a),  $\Delta$  decreases with increasing time. The rate of decrease of  $\Delta$  is specified the right hand side of equation (45), and can be zero only when either  $Q_j = Q_j^*$  or  $\dot{q}_j = \dot{q}_j^*$  at each point in the structure. Furthermore  $\Delta$  is a measure of the difference between the two velocity fields  $\dot{u}_i$  and  $\dot{u}_i^*$ , and is zero only when the two fields are identical. According to this measure, therefore, the two solutions approach each other as time progresses.

The convergence of the two solutions suggests a means of approximating unknown solution. If the response to the velocities  $v_i$  are not known, but the response to velocities  $v_i^*$  are known, we are assured that the difference between the response (as measured by  $\Delta$ ) will decrease with time. If the difference is initially not large i.e. if  $\Delta^0$  is not large, we may expect that the solutions will not be greatly different particularly away from the initial instants. Furthermore, we would not expect great differences in the velocities at any particular point on the structure (as opposed to integrated effect represented by  $\Delta$ ).

These ideas may be applied to attempts to approximate responses by means of mode solutions, as discussed in the previous section. However, in order to substitute a mode solution for  $\dot{u}_i^*$  in equation (45) it must be a complete solution to an initial value problem. The mode solutions used in the previous section were required to be kinematically admissible only. If it is possible to associate with the mode solution a statically admissible stress field which everywhere satisfies the yield condition, then the mode solution is a full solution to an initial value problem and may be substituted into equation (45).

If the mode solution is given as in equation (26),

$$\dot{u}_i^* (S, t) = v_i^* (S) T (t) \quad (48)$$

the accelerations are given by

$$\ddot{u}_i^* (S, t) = v_i^* (S) \frac{dT}{dt} (t) \quad (49)$$

$dT/dt$  may be found from equation (32). Further, if

$$v_i^* = \alpha \phi_i \quad (50)$$

as in equation (35), where  $\phi_i$  is a mode shape and  $\alpha$  is an arbitrary magnitude, it may be shown by substitution into equations (32) and (49) that  $\ddot{u}_i^* (S, t)$  is independent of  $\alpha$ . This means essentially that in the mode model the accelerations are independent of the initial velocity, i.e. the initial amplitude assigned to the mode. Using d'Alembert's principle and considering the inertia forces  $(-m\ddot{u}_i^*)$ , we have a static problem with tractions (in this case inertia forces) given on  $S_T$ , and displacements given zero on  $S_u$ . Using the limit theorems of Drucker, Greenberg and Prager<sup>20</sup>, if any distribution of stresses can be found which is in equilibrium with the inertia forces and which does not violate the yield condition, then there is associated with the mode solution a safe, statically admissible stress field. Again, this remains true for arbitrary  $\alpha$  in equation (50). It will be convenient to refer to a mode solution which can be associated with a safe statically admissible field as a safe mode solution.

If a solution of the form of (48) and (50) is to be substituted into (45), it is necessary to determine the initial mode amplitude  $\alpha$ . Since  $\Delta$  [equation (46)]

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<sup>20</sup> D. C. Drucker, H. J. Greenberg and W. Prager, "Extended limit design theory for continuous media," Quart. App. Math. 9, p. 381, 1952.

measures the difference between the solutions, it is logical to choose  $\alpha$  such that  $\Delta^0$  is as small as possible. This may be written as

$$\frac{d}{d\alpha} \int_S \frac{m}{2} (v_i - \alpha \phi_i) (v_i - \alpha \phi_i) dS = 0 \quad ($$

Differentiating and solving for  $\alpha$ , we obtain

$$\alpha = \frac{\int_S m v_i \phi_i dS}{\int_S m \phi_i \phi_i dS} \quad ($$

Comparison with equation (39) shows that the result is identical to that obtained earlier by a different argument. Multiplying through by  $\alpha$ , we see that, using (50),

$$\int_S m \alpha \phi_i \alpha \phi_i dS = \int_S m v_i^* v_i^* dS = \int_S m v_i v_i^* dS$$

or

$$\int_S m (v_i - v_i^*) v_i^* dS = 0$$

Equation (53a) may be used to show that, if (52) is satisfied,

$$\Delta^0 = \int_S \frac{m}{2} v_i v_i dS - \int_S \frac{m}{2} v_i^* v_i^* dS$$

i. e. the initial measure of difference is simply the initial difference between the energies of the solutions. Equation (53b) shows that the difference between  $v_i$  and  $v_i^*$  is orthogonal to  $v_i^*$ ; this result is suggestive of elastic normal mode analysis, considering  $v_i^*$  to be analogous to any one mode.

This discussion indicates that the choice of a safe mode solution to approximate the unknown solution leads to initial conditions identical to those found earlier, and further to an assurance that a measure of the differences between the solutions must decrease. If the initial difference is small, i. e. small compared to the initial energy in any one solution, we are provided with a reliable check of the validity of the one degree of freedom model. No arbitrary assumptions, other than the choice of a mode shape, need be made.

One further useful piece of information may be added. From a proposition due to Martin<sup>21</sup>, the time  $t$  at which the velocities  $\dot{u}_i$  vanish in the real solution can be written

$$t_f \geq \frac{\int_S m v_i v_i^* dS}{\int_S \bar{D} dS} \quad (55)$$

where  $v_i^*$  is the initial mode velocity and  $\bar{D}$  is the initial rate of energy dissipation in the mode. (This proposition does not require that the mode be a safe mode.) However, from (34), (32) and (53a),

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<sup>21</sup>J. B. Martin, "Impulsive loading theorems for rigid plastic continua," Proc. Eng. Mech. Div., A.S.C.E., 90 (EM5), p.27, 1964.

$$t_f^* = \frac{\int_S m v_i^* v_i^* dS}{\int_S \bar{D} ds} = \frac{\int_S m v_i v_i^* dS}{\int_S \bar{D} ds} \quad (56)$$

provided that the initial amplitude of the mode satisfies the optimum requirement that  $\Delta^0$  be a minimum. Hence

$$t_f \geq t_f^* \quad (57)$$

i. e. the velocities in the model vanish before the velocities in the actual solution. Experience shows that  $t_f$  and  $t_f^*$  are very often equal, and in most cases the difference is small.

In the following section we shall discuss the application of this method to illustrative examples.

## 5. EXAMPLES

In order to demonstrate the effectiveness of the approximation using the mode solution we shall compare the results of the solutions discussed in § 2 with their approximations.

(i) Consider first the clamped beam shown in Figs. 6(a) and 9(a). The final central displacement is given in equations (17) and (18), and is a function of  $b/l$ . A mode solution has been found for this problem, Fig. 9(c), and the final displacement is given in equation (44). It remains to check that the mode chosen is indeed a safe mode, and to find the relative value of the initial difference  $\Delta^0$ .

The acceleration of the central point in the beam in the mode solution may be easily calculated from the initial central velocity [equation (42)] and the time of duration [equation (43)] since it is known that the velocity-time relation is linear.

This is given by

$$\ddot{u}^* = - \frac{24M_0}{ml^2} \quad (5^0)$$

Using d'Alembert's principle, the loading diagram for the beam is shown in Fig. 11(a) and the bending moment diagram is shown in Fig. 11(b). It is readily shown that the maximum bending moment is  $M_0$ , and that it always occurs in the center of the beam. Hence the mode is a safe mode solution. In order to compute  $\Delta^0$  we make use of equation (54). A relative measure of the magnitude of  $\Delta^0$  may be obtained by dividing  $\Delta^0$  by the initial energy of the actual problem. Let this be  $K^0$ . Then

$$K^0 = \frac{1}{2} m b v_0^2 \quad (59a)$$

$$\begin{aligned} \Delta^0 &= \frac{1}{2} m b v_0^2 - 2 \int_0^{l/2} \frac{m}{2} \dot{u}_0^2 dx \\ &= \frac{1}{2} m b v_0^2 - \frac{3}{8} m l v_0^2 \left[ \frac{2b}{l} - \left( \frac{b}{l} \right)^2 \right]^2 \end{aligned} \quad (59b)$$

$$\frac{\Delta^0}{K^0} = 1 - \frac{3}{4} \frac{b}{l} \left( 2 - \frac{b}{l} \right)^2 \quad (59c)$$

The final central displacement computations have been plotted in Fig. 12. In addition to the actual and mode solutions, the upper bound which may be computed by a method proposed by Martin<sup>21</sup> is given. In Fig. 13 the mode solution error and the ratio  $\Delta^0/K^0$  are plotted on the same figure. This diagram shows that the error is small when  $\Delta^0/K^0$  is small, and that the error increases as  $\Delta^0/K^0$  increases. It can be expected that the error is zero when  $\Delta^0/K^0$  is a minimum, and indeed this is so.

Figs. 14 and 15 give further details for the case of  $b/l = 1$ . Fig. 14 gives the central velocity-time curves for the actual solution and the mode solution. It can be seen that the velocities coincide after one third of the total deformation time has elapsed. In fact the velocities coincide everywhere on the beam during this period and  $\Delta$  [equation (46b)] is zero. Thus  $\Delta$  decreases rapidly (and in this case vanishes) even though  $\Delta^0/K^0$  (Fig. 13) has an appreciable magnitude.

Fig. 15 compares the final shapes of the deformed beams for the case  $b/l = 1$ , showing that the displacements in the mode solution overestimate the actual displacements in some regions, and underestimate them in others.

A mode solution for the pinned end case of the beam discussed above (Fig. 7a) may be obtained in a manner identical to that for the clamped case. The mode shape is taken to be that in Fig. 9(c). The initial amplitude is not affected by the change in boundary conditions, but the deformation time  $t_f$  is simply doubled. It may readily be shown that this is a safe mode solution. A comparison of the approximate solution with the actual solution for the central displacement<sup>19</sup> for two values of  $b/l$  is given in Fig. 16.

(ii) Consider secondly the cantilever with an attached tip mass shown in Fig. 4. The solution, giving the tip mass velocity as a function of time, appears in equations (8) and (10). The final tip mass displacement is given in equation (11).

In this case a good guess at a mode solution is clearly a velocity field involving a rigid body rotation about the support, shown in Fig. 17. The velocity field and rotation rate at the base may be written in terms of  $\dot{z}^*$ , the velocity of the tip mass. From equations (28) and (32), the acceleration of the tip mass will be

$$\ddot{z}^* = - \frac{\dot{z}^* (M_o \dot{z}^* / l)}{G (\dot{z}^*)^2 + \int_0^l m \left( \frac{x}{l} \dot{z}^* \right)^2 dx} = - \frac{M_o}{G l \left( 1 + \frac{2}{3} \gamma \right)} \quad (60)$$

where  $\gamma = ml/2G$  as before. The bending moment diagram may now be drawn, and is shown in Fig. 18. It may readily be seen that the bending moment does not exceed



$M_0$  at any point for any value of  $\gamma$ . Hence the chosen approximation is a safe mode solution.

The optimum initial value of the tip velocity  $\dot{z}^*$  may be found as before. The actual initial velocity is zero except at the tip where it has value  $v$ . If  $v^*$  is the initial value of  $\dot{z}^*$ , we require, from equation (53a),

$$Gv^*v = G(v^*)^2 + \int_0^l m \left( \frac{x}{l} v^* \right)^2 dx$$

$$\frac{v^*}{v} = \frac{1}{1 + \frac{2}{3} \gamma} \quad (61)$$

The tip mass velocity is then given by

$$\frac{\dot{z}^*}{v} = \frac{1}{1 + \frac{2}{3} \gamma} \left( 1 - \frac{t}{t_f^*} \right) \quad (62)$$

where, from equation (56)

$$t_f^* = \frac{Gv\dot{z}^*}{M_0 \left( \frac{\dot{z}^*}{l} \right)} = \frac{Gl v}{M_0}$$

Fig. 19 shows the tip mass velocities plotted for the actual solution [equations (8) and (10)] and for the mode solution, for the particular case  $\gamma = 1$ . In this case it is seen that the velocities are the same for  $M_0 t / Gl v \geq 0.167$ ; it may readily be shown that  $\Delta$  vanishes at time  $M_0 t / Gl v = 0.167$ .

The final tip displacement is a function of the parameter  $\gamma$ , and hence the accuracy of the approximation will depend on  $\gamma$ . The final tip displacements have been plotted as a function of  $\gamma$  in Fig. 20, and show that the discrepancy varies from zero at  $\gamma = 0$  to an underestimate of about 15% at  $\gamma = 3$ . On the same figure  $\Delta^0/K^0$  has been plotted. When  $\gamma = 3$   $\Delta^0/K^0$  has the value 0.67. Despite this large value of  $\Delta^0/K^0$  the approximation is fairly good.

One further example is summarized in Fig. 21. In this case the energy ratio  $\Delta^0/K^0$  is extremely small. Considering the areas under the curves to obtain displacements at the center of the beams, it can be seen that the difference between the actual solution and the mode solution is extremely small.

## 6. CONCLUSIONS

In this paper an attempt has been made to rationalize the setting up of a one degree of freedom approximation in elementary rigid-plastic theory for imp loading. The method requires that a mode shape be chosen; thereafter, the deceleration (or the equivalent spring force and mass) and the initial mode velocity follow without further assumptions. The concept of a safe mode and the analysis given in Section 4 provide a criterion by which good mode approximations may be recognized. For illustrative purposes the method has been applied to extremely simple examples. There is no conceptual difficulty in applying the technique to more complicated cases.

It is seen that even in cases where  $\Delta^0/K^0$  is fairly large the approximation can be reasonably good. Only where  $\Delta^0/K^0$  is small, however, can it be taken that the approximation will certainly be good. The other cases emphasize that good approximations can be obtained when  $\Delta^0/K^0$  is large, as undoubtedly good approximation could be obtained with mode solutions which were not safe in the sense used in this paper. This paper shows only that a certain limited class of mode approximations can be considered reliable.

The technique described could be applied with changes, to problems with time dependent loading and to certain other viscous type material idealizations such as rigid-visco-plastic. Attempts are being made to develop useful approximating techniques for these cases.

## ACKNOWLEDGEMENT

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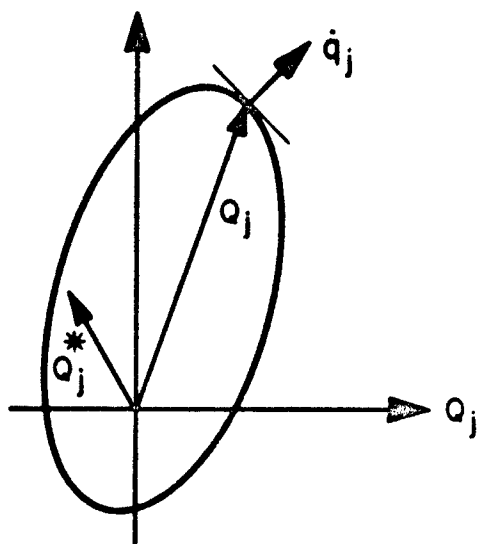


FIGURE 1

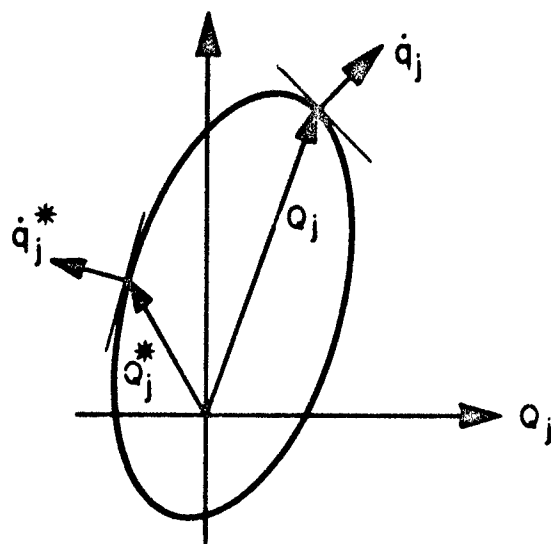


FIGURE 2

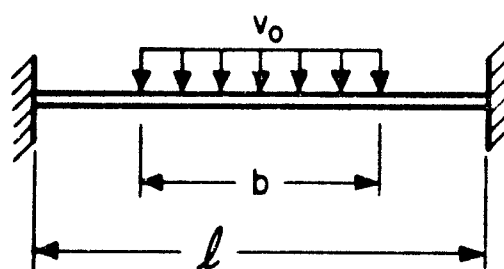


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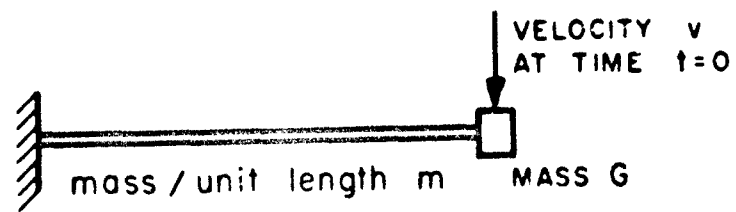
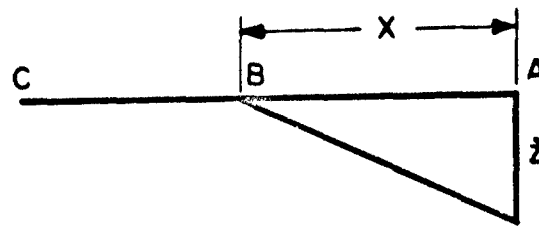
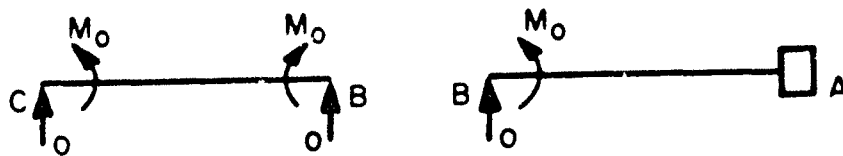


FIGURE 4



VELOCITY DIAGRAM

(a)



FREE BODY DIAGRAMS

(b)

FIGURE 5

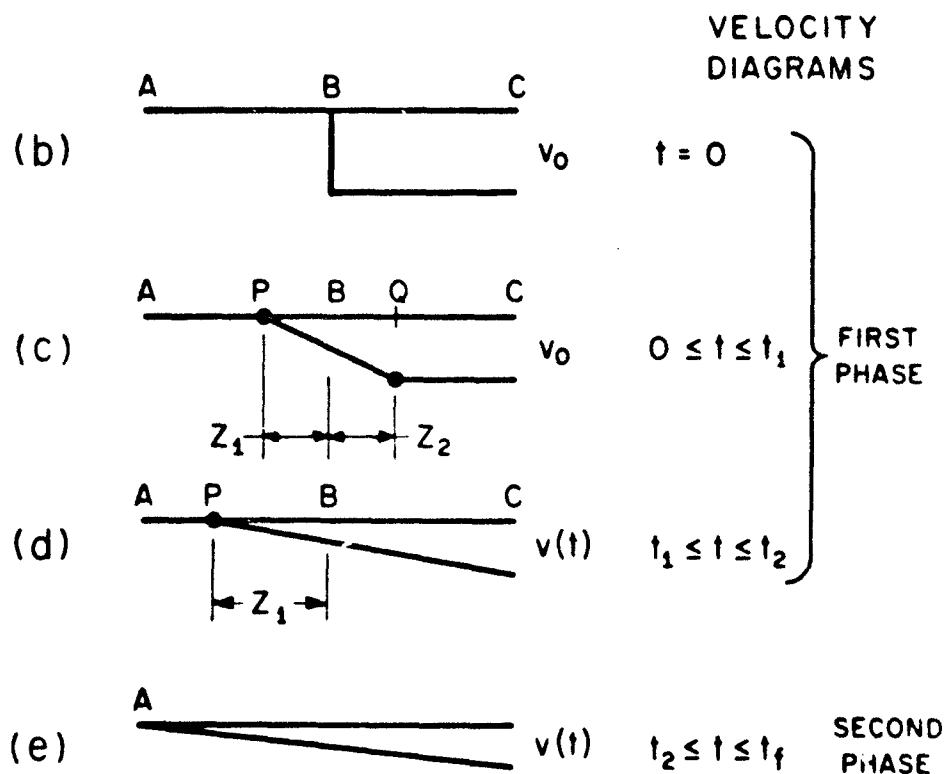
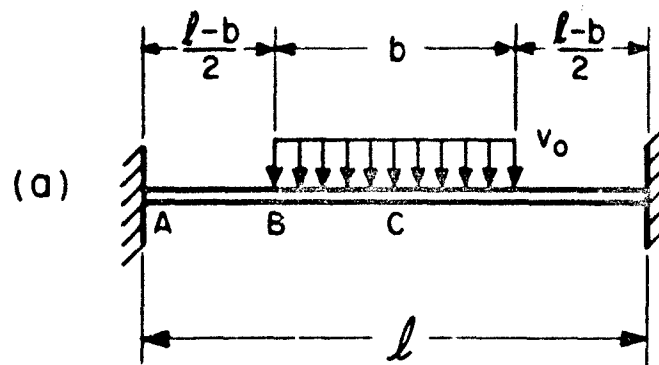


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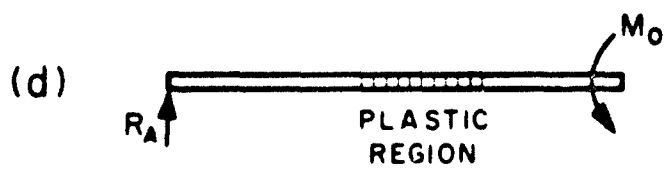
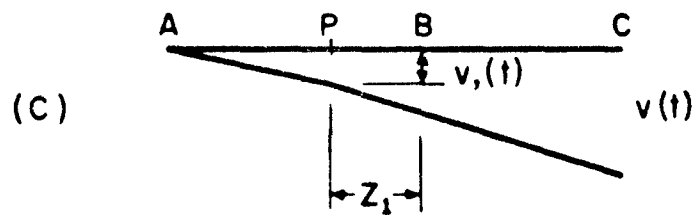
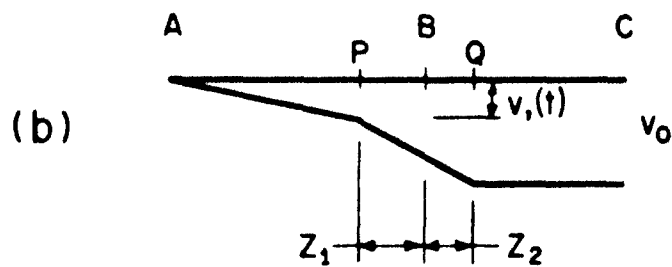
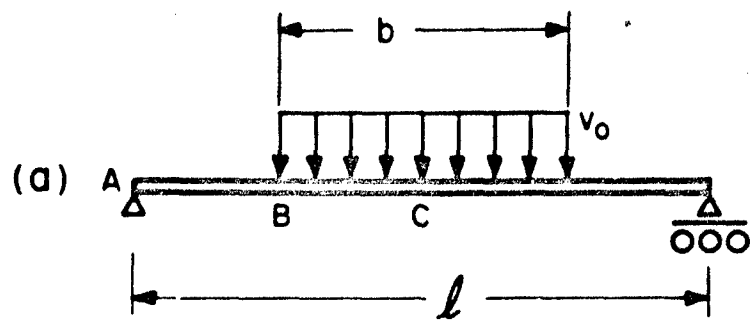


FIGURE 7



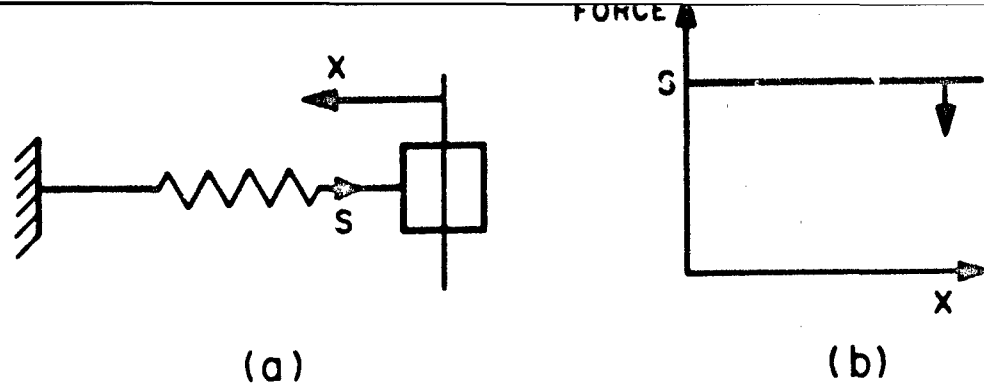


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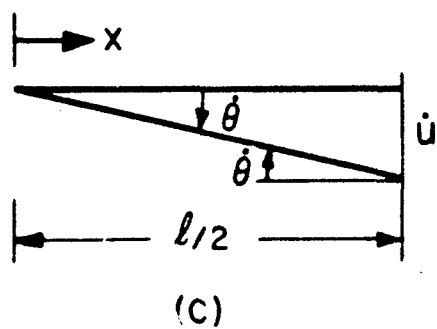
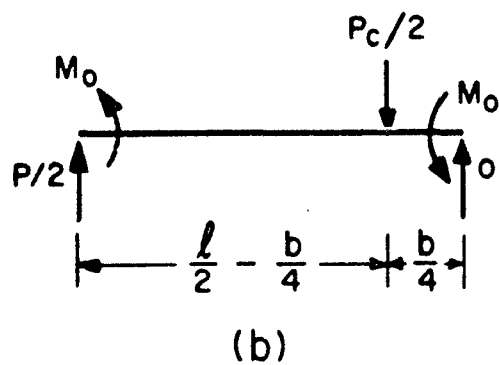
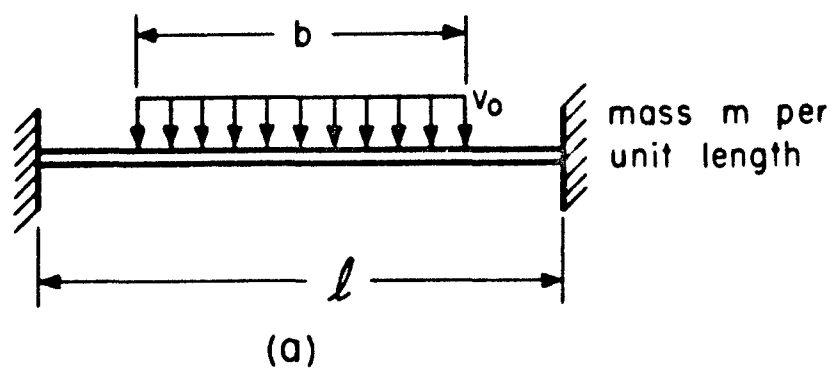


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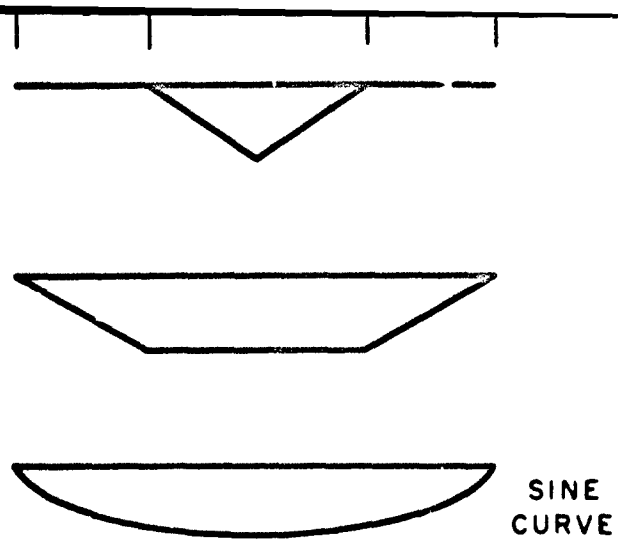


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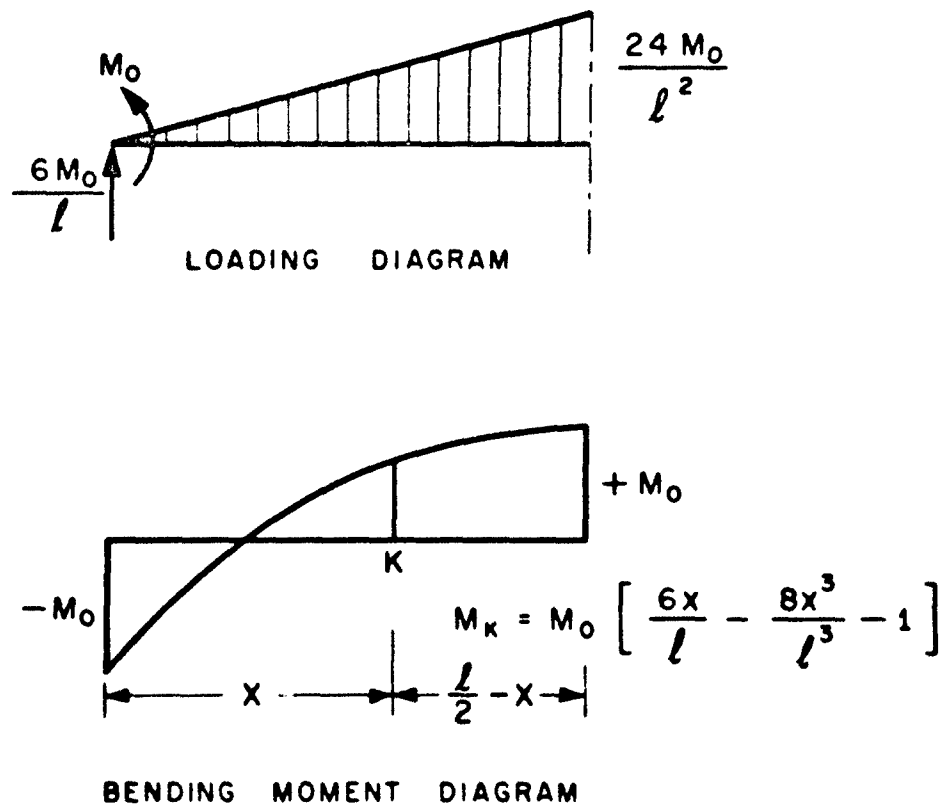


FIGURE 11

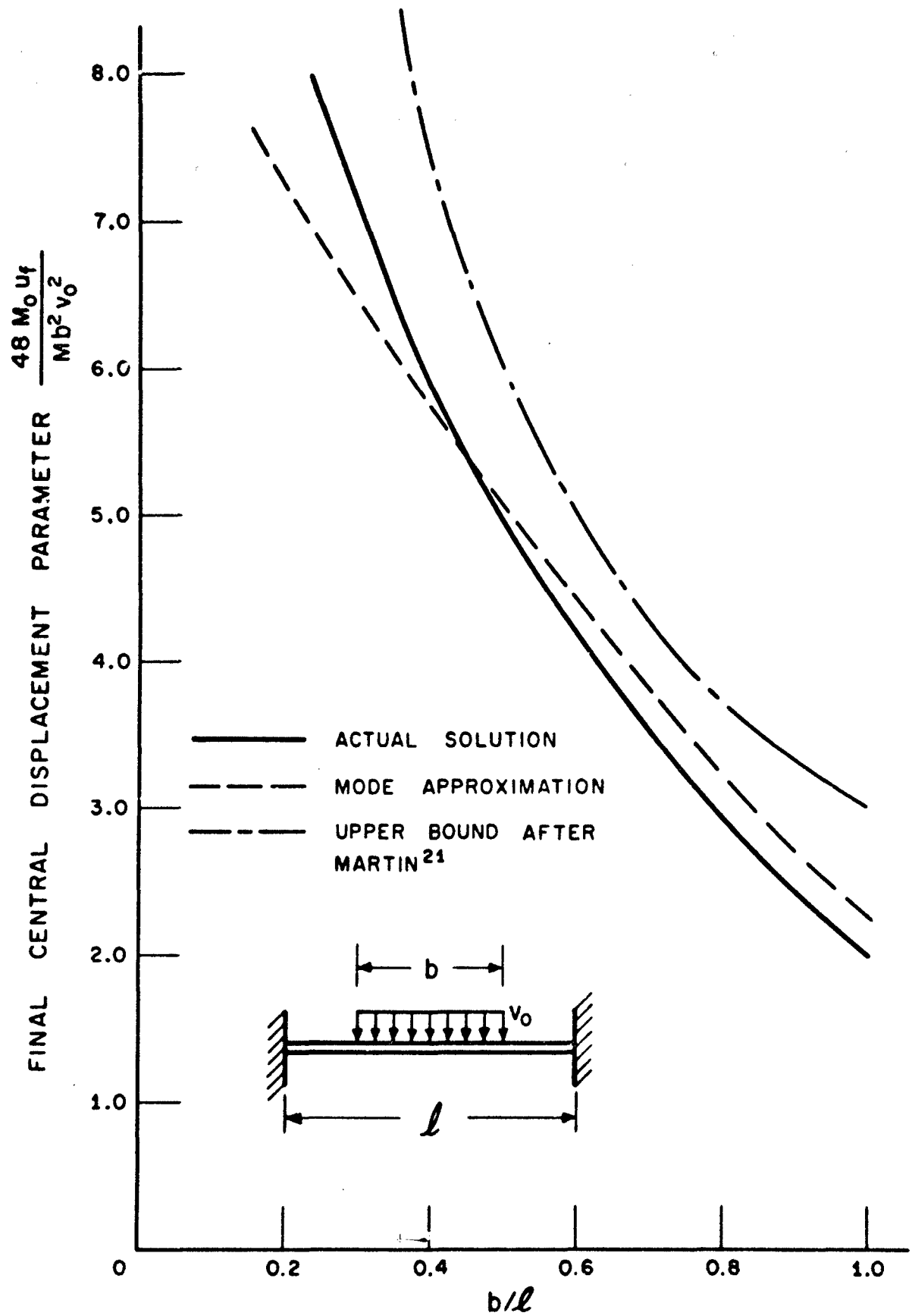


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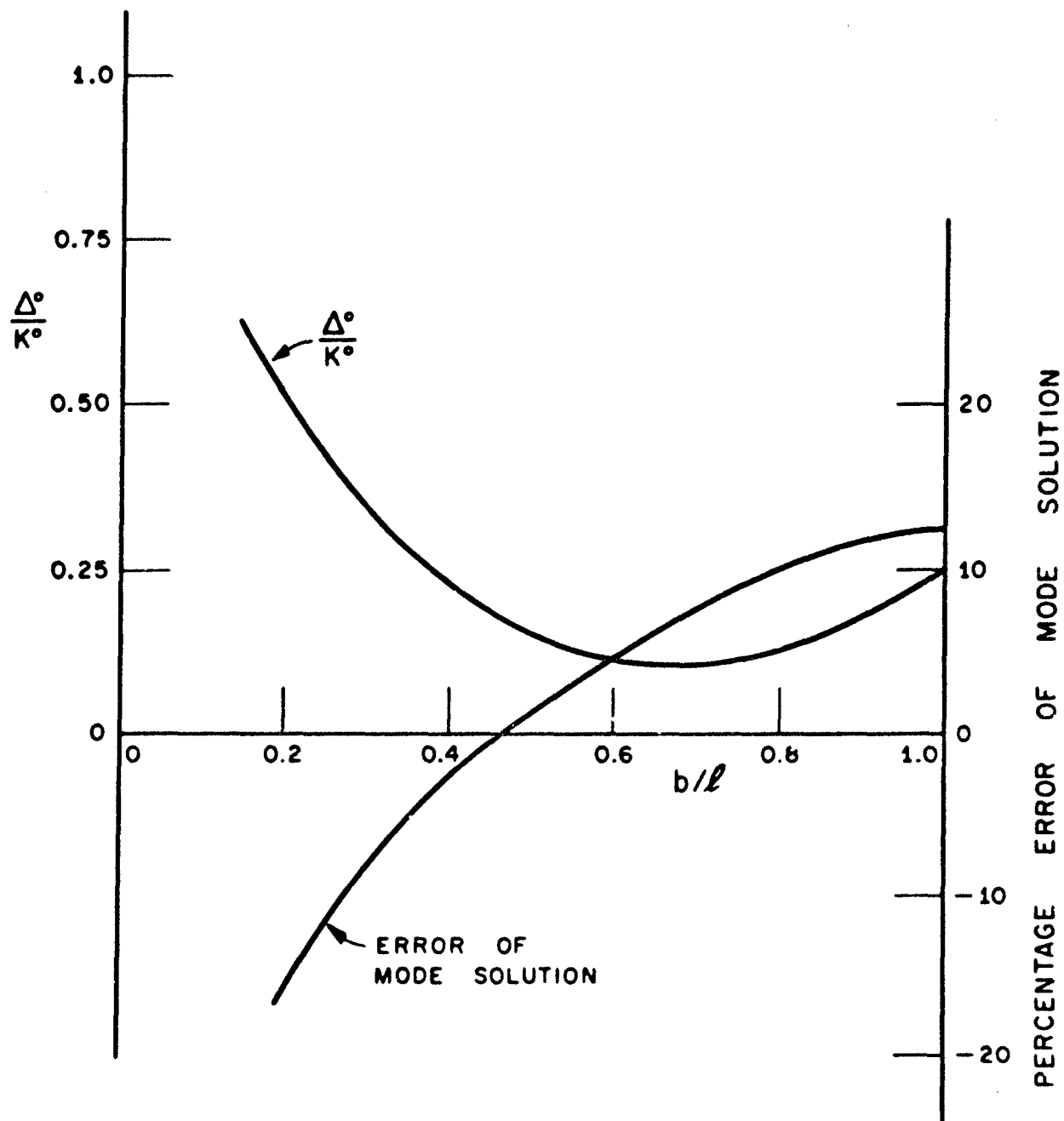


FIGURE 13

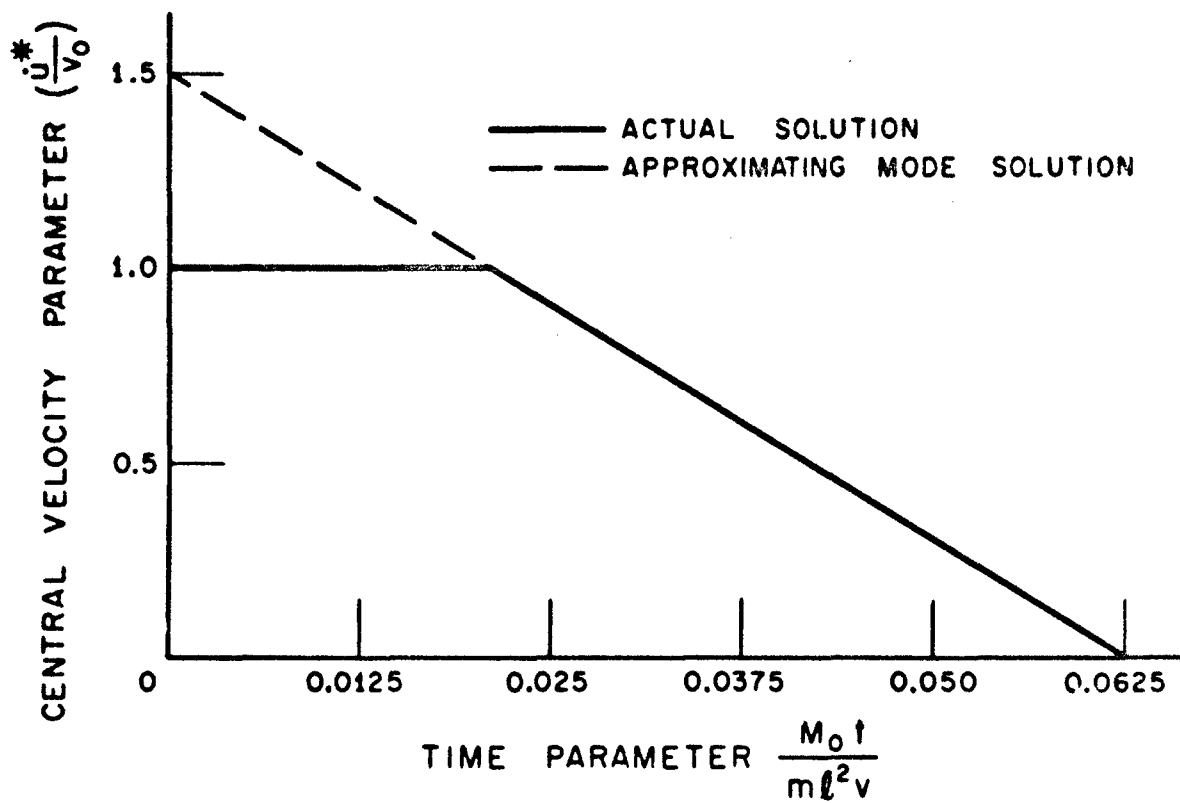


FIGURE 14

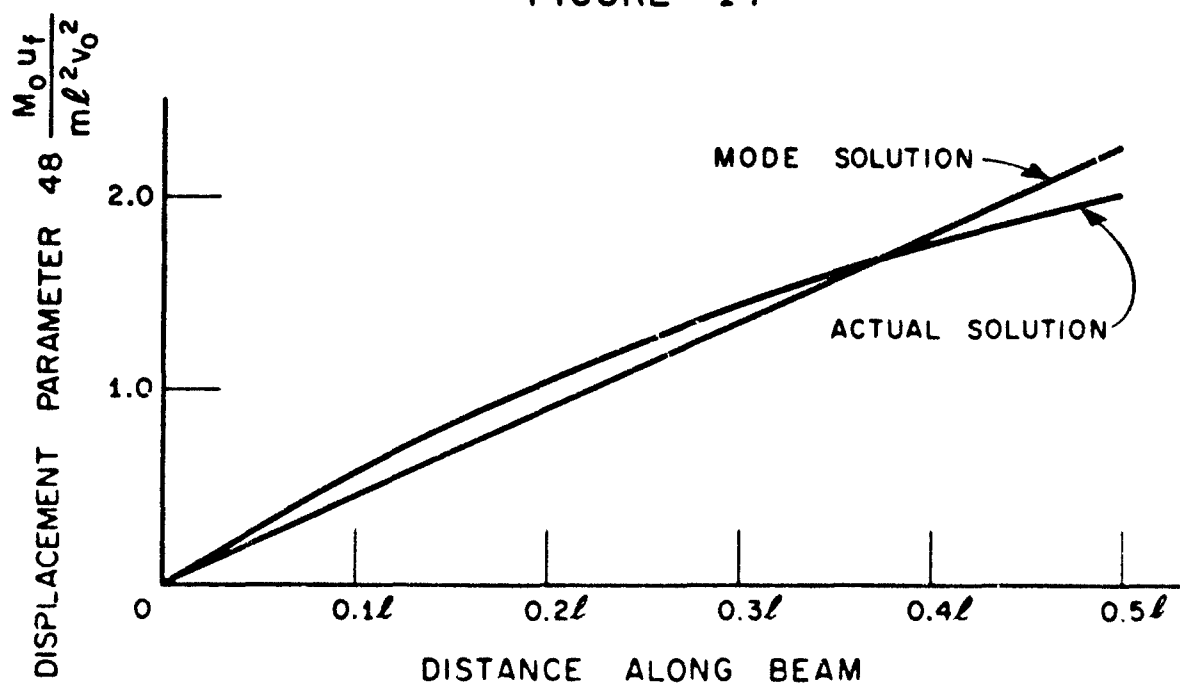


FIGURE 15

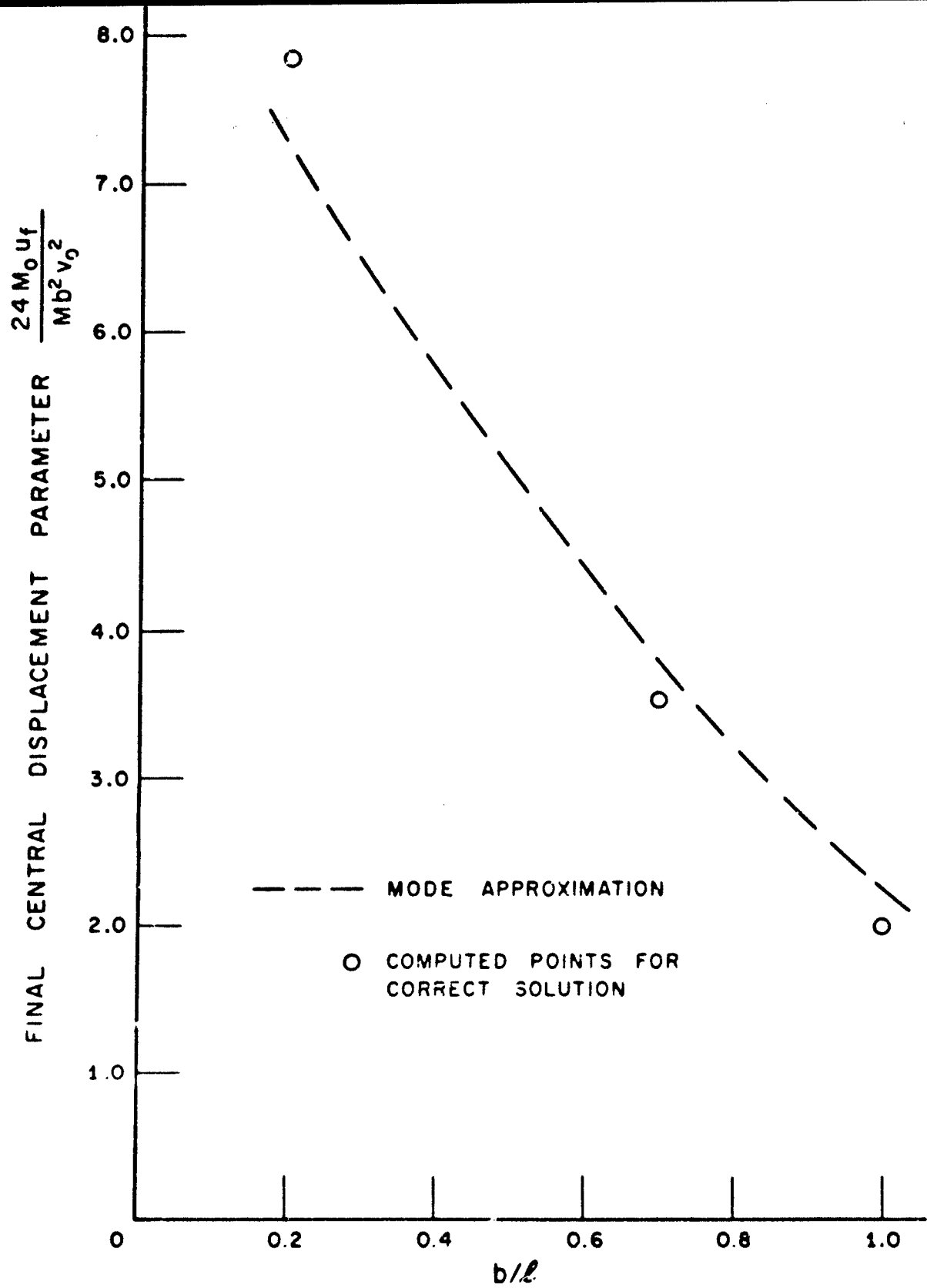


FIGURE 16

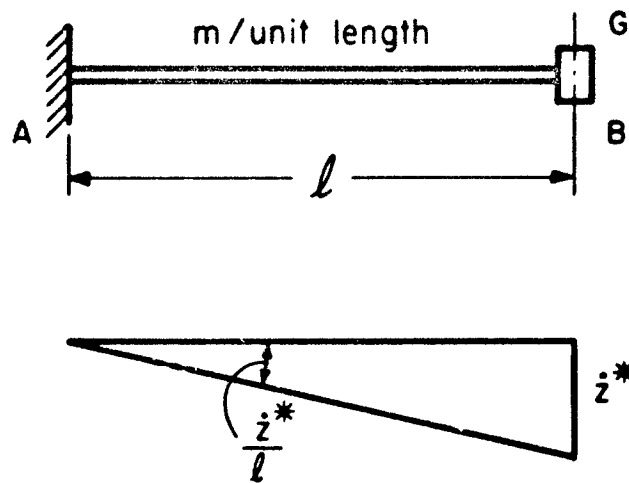


FIGURE 17

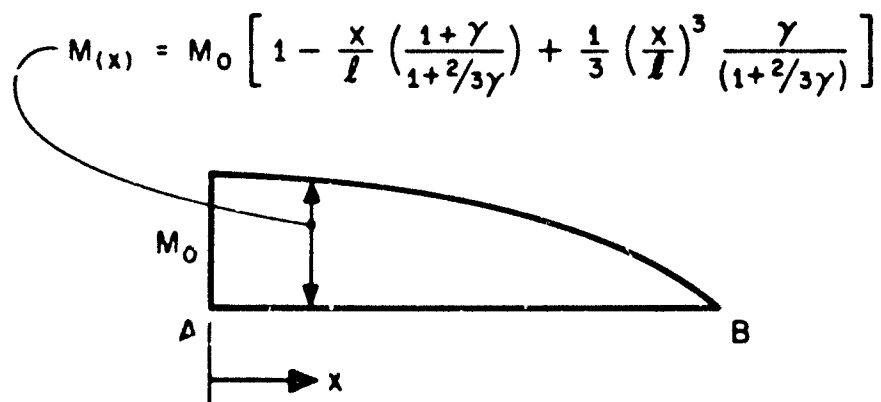


FIGURE 18

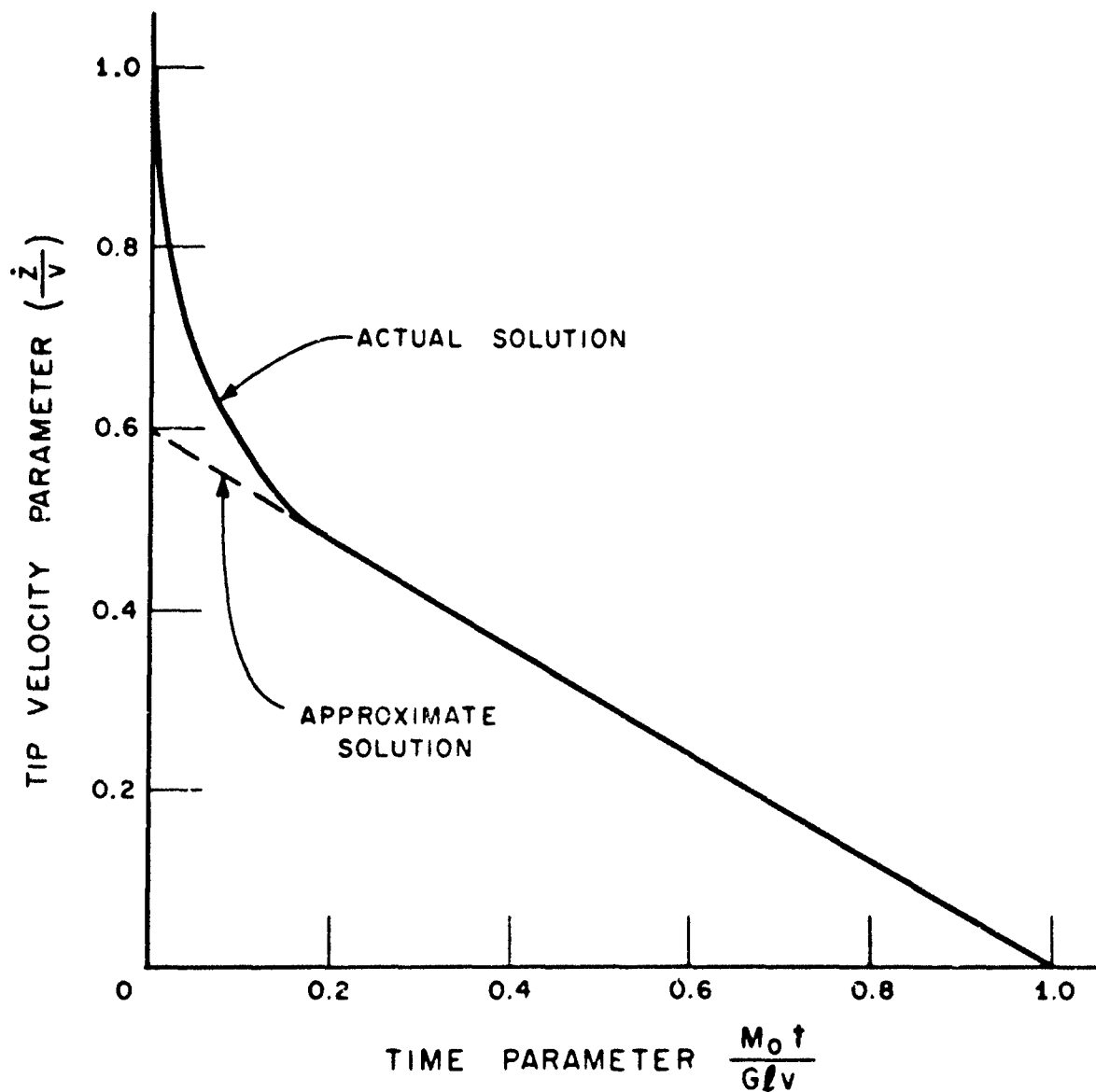


FIGURE 19 COMPARISON OF CORRECT AND APPROXIMATE  
TIP VELOCITY-TIME CURVES FOR  
IMPULSIVELY LOADED CANTILEVER  
WITH  $\gamma = 1$



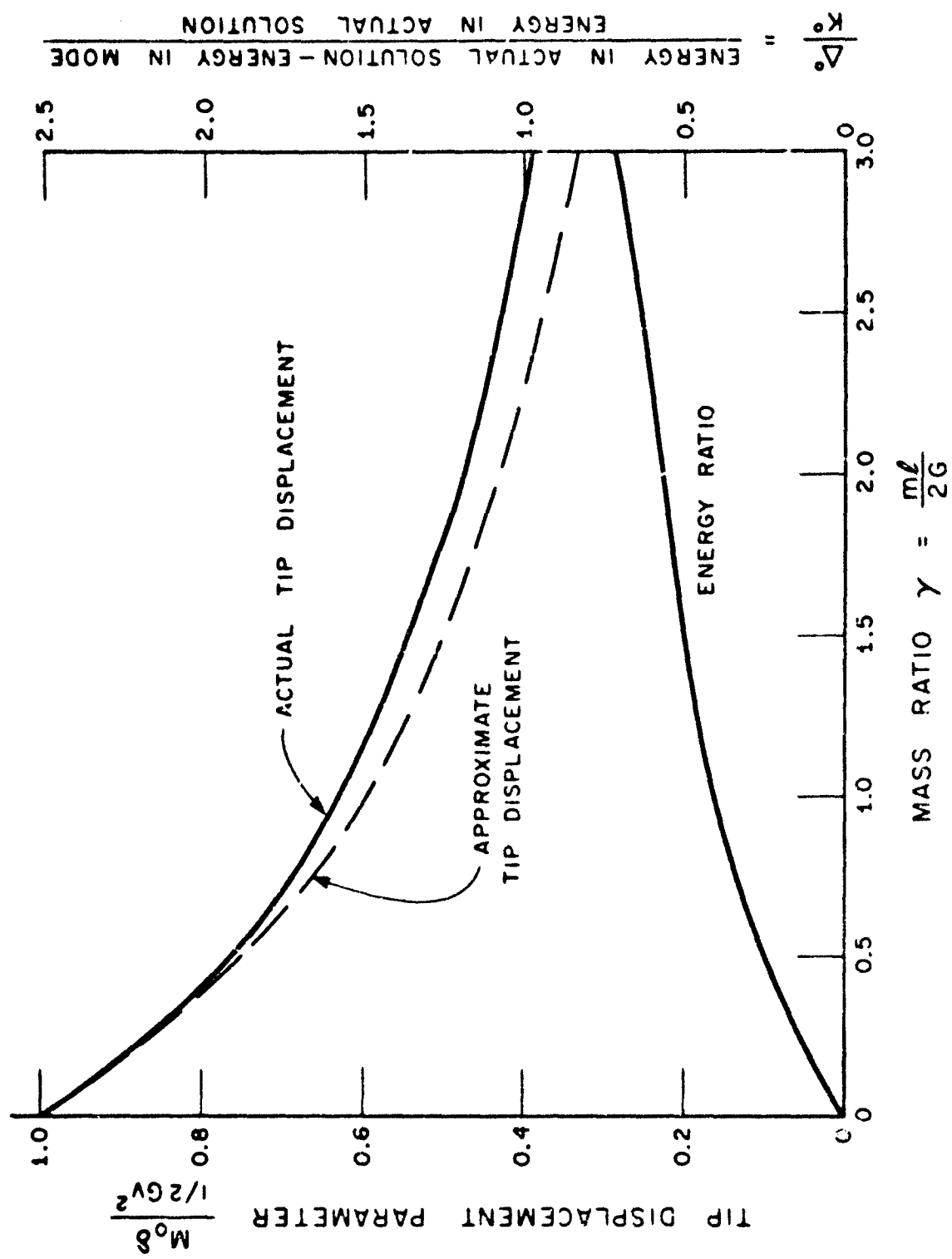
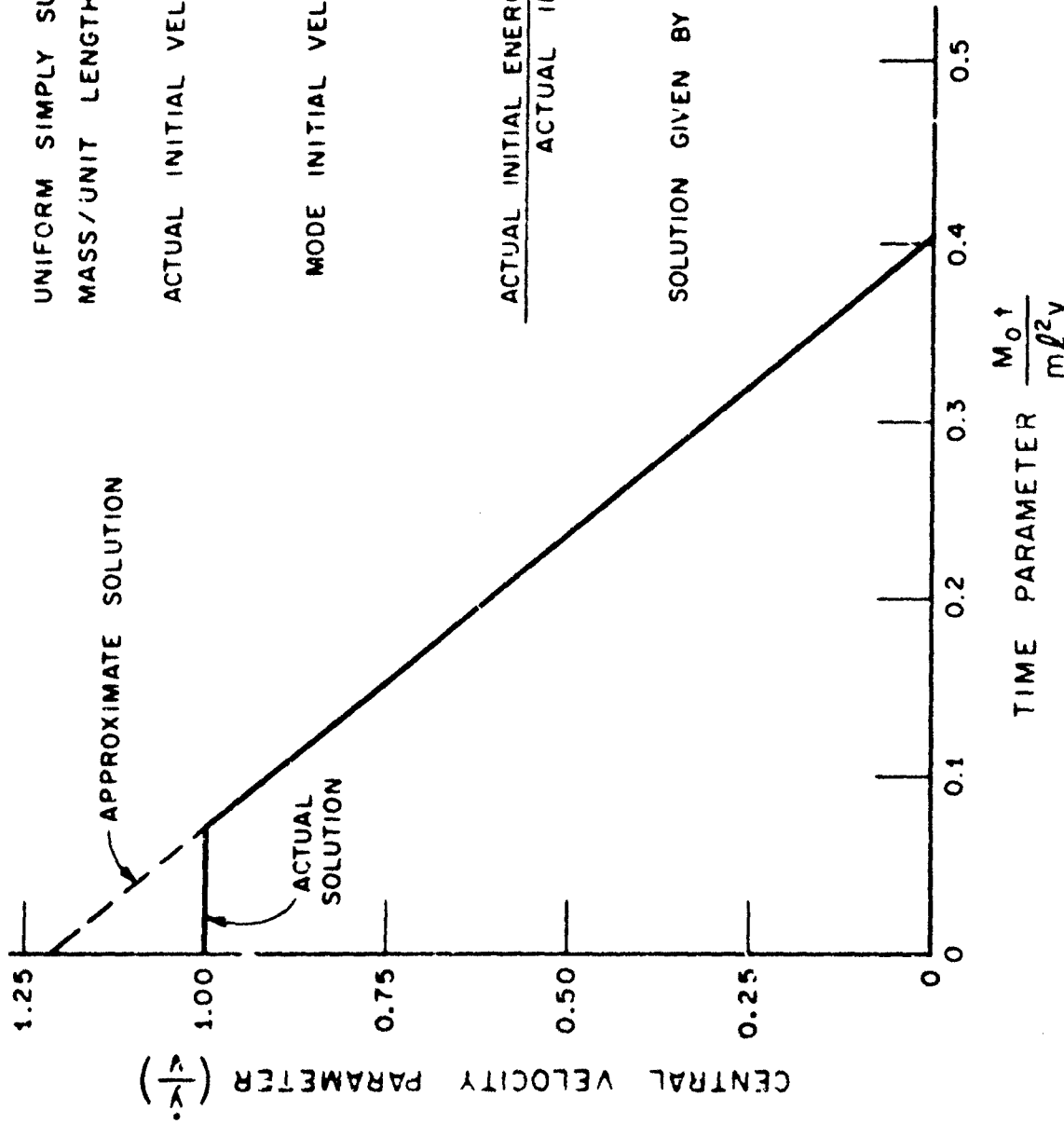


FIGURE 20 COMPARISON OF CORRECT AND APPROXIMATE TIP DISPLACEMENTS FOR IMPULSIVELY LOADED CANTILEVER



UNIFORM SIMPLY SUPPORTED BEAM, LIMIT MOMENT  $M_0$ ,  
MASS/UNIT LENGTH  $m$ , SPAN  $l$

ACTUAL INITIAL VELOCITY DISTRIBUTION

$$\dot{u} = v_0 \sin \frac{\pi x}{l}$$

MODE INITIAL VELOCITY DISTRIBUTION

$$y^{*0} = \frac{12}{\pi^2} v_0$$

$$\frac{\text{ACTUAL INITIAL ENERGY} - \text{MODE INITIAL ENERGY}}{\text{ACTUAL INITIAL ENERGY}} = 0.016$$

SOLUTION GIVEN BY SEILER, COTTER AND SYMONDS<sup>4</sup>

FIGURE 21 APPROXIMATE SOLUTION FOR UNIFORM SIMPLY SUPPORTED BEAM WITH SINUSOIDAL IMPULSE

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13. ABSTRACT <p>An attempt is made to provide a rational method of constructing one degree of freedom approximations for impulsively loaded metal structures which are analysed according to an elementary rigid-plastic theory. The approximation follows automatically from a chosen mode shape, and a criterion for determining good mode shapes is introduced.</p>			

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